☆ Invariants & Lagrangians

Ex $SO(2)$

- Vectors $\vec{a}$, $\vec{b}$

- Invariants $\Longrightarrow$

$$\boxed{\vec{a}\cdot\vec{a}\ ,\quad \vec{b}\cdot\vec{b}\ ,\quad \vec{a}\cdot\vec{b}\ ,\quad a_i\, b_j\, \varepsilon_{ij}}$$

Representations of Lorentz group

- $\left(\frac{1}{2}, \frac{1}{2}\right) \equiv$ defining representation $\quad \Lambda = \underbrace{e^{-i\vec{\theta}\cdot\vec{J} + i\vec{\eta}\cdot\vec{K}}}_{4\times 4}$

- $\left(\frac{1}{2}, 0\right) \quad \Lambda_L = e^{-\frac{1}{2}(i\vec{\theta}+\vec{\eta})\cdot\vec{\sigma}}$

- $\left(0, \frac{1}{2}\right) \quad \Lambda_R = e^{-\frac{1}{2}(i\vec{\theta}-\vec{\eta})\cdot\vec{\sigma}}$

- $\Lambda_L^\dagger \Lambda_R = \mathbb{1}$ $\qquad$ $\Lambda_R^\dagger \Lambda_L = \mathbb{1}$

- $\partial \equiv \sigma^\mu \partial_\mu$ $\qquad$ $\bar{\partial} \equiv \bar{\sigma}^\mu \partial_\mu$

- $\sigma^\mu \equiv (\mathbb{1}, \sigma^i)$ $\qquad$ $\bar{\sigma}^\mu \equiv (\mathbb{1}, -\sigma^i)$

$\bar{\sigma}^\mu = P^\mu{}_\nu \sigma^\nu$

$P^\mu{}_\nu = (1, -1, -1, -1)$

$x'_P = Px$

- v^μ :

$v \equiv v^\mu \sigma_\mu$

$\Lambda_L(\theta,\eta)\, v\, \Lambda_L^\dagger(\theta,\eta) =$

$= \left(\Lambda^\mu{}_\nu(\theta,\eta)\, v^\nu\right) \sigma_\mu$

$\bar{v} \equiv v^\mu \bar{\sigma}_\mu$

$\Lambda_R \bar{v} \Lambda_R^\dagger =$

$= \left(\Lambda^\mu{}_\nu v^\nu\right) \bar{\sigma}_\mu$

- $\psi_L'(x') = \Lambda_L \psi_L(x)$ $\qquad$ $\bar\partial' \psi_L'(x') = \Lambda_R \bar\partial \psi_L(x)$

  $\psi_R'(x') = \Lambda_R \psi_R(x)$ $\qquad$ $\partial' \psi_R'(x') = \Lambda_L \partial \psi_R(x)$

- $\psi_L'(x')^\dagger = \psi_L(x)^\dagger \Lambda_L^\dagger = \psi_L^\dagger \Lambda_R^{-1}$

  $\psi_R'(x')^\dagger = \psi_R(x)^\dagger \Lambda_R^\dagger = \psi_R^\dagger \Lambda_L^{-1}$

$\Longrightarrow$

▲ Weyl $\qquad \mathcal{L} = i \psi_L^\dagger \bar\partial \psi_L = i \psi_L'^\dagger \bar\partial' \psi_L'$

▲ Dirac $\qquad \mathcal{L} = i \psi_L^\dagger \bar\partial \psi_L + i \psi_R^\dagger \partial \psi_R - m \psi_L^\dagger \psi_R - m \psi_R^\dagger \psi_L$

- Weyl

$$S = \int d^4x \; i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$$

like in QM

$-i\vec{\nabla} \equiv$ hermitean

$$S^* = \int d^4x \; -i \partial_\mu \psi^\dagger \bar{\sigma}^{\mu\dagger} \psi_L$$

$$= \int d^4x \left[-i \partial_\mu \cancel{\left(\psi^\dagger \bar{\sigma}^\mu \psi_L \right)} + i \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi \right]$$

$$\frac{\delta S}{\delta \psi^\dagger_{L\alpha}} = \frac{\partial \mathcal{L}}{\partial \psi^\dagger_{L\alpha}} = i \left(\bar{\partial} \psi_L \right)_\alpha \xrightarrow{\text{eqs motion}} i \bar{\partial} \psi_L = 0$$

Weyl eq.

- <u>Dirac</u>

$$\psi_L \to e^{i\alpha}\psi_L$$
$$\psi_R \to e^{-i\alpha}\psi_R$$

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m\psi_L^\dagger \psi_R - m\psi_R^\dagger \psi_L$$

<u>def</u> $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\qquad \bar{\Psi} = \psi^\dagger \gamma^0 = (\psi_R^\dagger, \psi_L^\dagger)$

$$\not{\partial} \equiv \gamma^\mu \partial_\mu \qquad \gamma^\mu \doteq \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\Rightarrow \mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi \Longrightarrow \frac{\delta S}{\delta \bar{\Psi}} = (i\not{\partial} - m)\psi = 0$$

Lorentz: $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \longrightarrow \begin{pmatrix} \Lambda_L \psi_L \\ \Lambda_R \psi_R \end{pmatrix} = \underbrace{\begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}}_{\equiv \Lambda_D} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$\bar{\Psi} = \left(\psi_R^+ \;\; \psi_L^+ \right) \longrightarrow \left(\psi_R^+ \;\; \psi_L^+ \right) \underbrace{\begin{pmatrix} \Lambda_R^+ & 0 \\ 0 & \Lambda_L^+ \end{pmatrix}}_{\equiv \Lambda_D^{-1}}$

- $\bar{\Psi} \Psi \longrightarrow \bar{\Psi} \Lambda_D^{-1} \Lambda_D \Psi = \bar{\Psi} \Psi$

- $\bar{\psi}\gamma^\mu\psi \to \bar{\psi}\Lambda_D^{-1}\gamma^\mu\Lambda_D\psi = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$
- $\bar{\psi}\gamma^\mu\gamma^\nu\psi \to \bar{\psi}\Lambda_D^{-1}\gamma^\mu\Lambda_D\Lambda_D^{-1}\gamma^\nu\Lambda_D\psi = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}\gamma^\rho\gamma^\sigma\psi$
- etc.

Majorana : $i\bar{\partial}\psi_L - m\varepsilon\psi_L^* = 0$

- $\mathcal{L}$?

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \frac{m}{2}\psi_L^\dagger \varepsilon \psi_L^* + \frac{m}{2}\psi_L^T \varepsilon^T \psi_L$$

$$\frac{m}{2}\psi_\alpha^* \varepsilon_{\alpha\beta} \psi_\beta^*$$

$\psi_L = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ $\psi_L^* = \begin{pmatrix}\psi_1^* \\ \psi_2^*\end{pmatrix}$

$\psi_L^\dagger = (\psi_1^* \; \psi_2^*)$

$$\frac{m}{2}\varepsilon_{\alpha\beta}\psi_\alpha^*\psi_\beta^*$$

$$A_{\alpha\beta} \quad S_{\alpha\beta}$$

General solution of Dirac equation

$$\boxed{(i\not{\partial} - m)\psi(x) = 0}$$

- Fourier space $\quad \hat{\psi}(p) = \int e^{ipx}\psi(x)\, d^4x \qquad px \equiv p_\mu x^\mu$

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \hat{\psi}(p)$$

$$(i\not{\partial} - m)\psi = \int \frac{d^4p}{(2\pi)^4} (\not{p} - m)\hat{\psi}(p)\, e^{-ipx} = 0$$

$$\boxed{(\not{p} - m)\hat{\psi}(p) = 0}$$

- $$\left(\not{p} - m\right)_{4\times 4} \quad \hat{\psi}(\hat{p})_{4\times 1} = 0 \qquad (\not{p} - m) \to \Lambda_D^{-1}(\not{p}-m)\Lambda_D$$

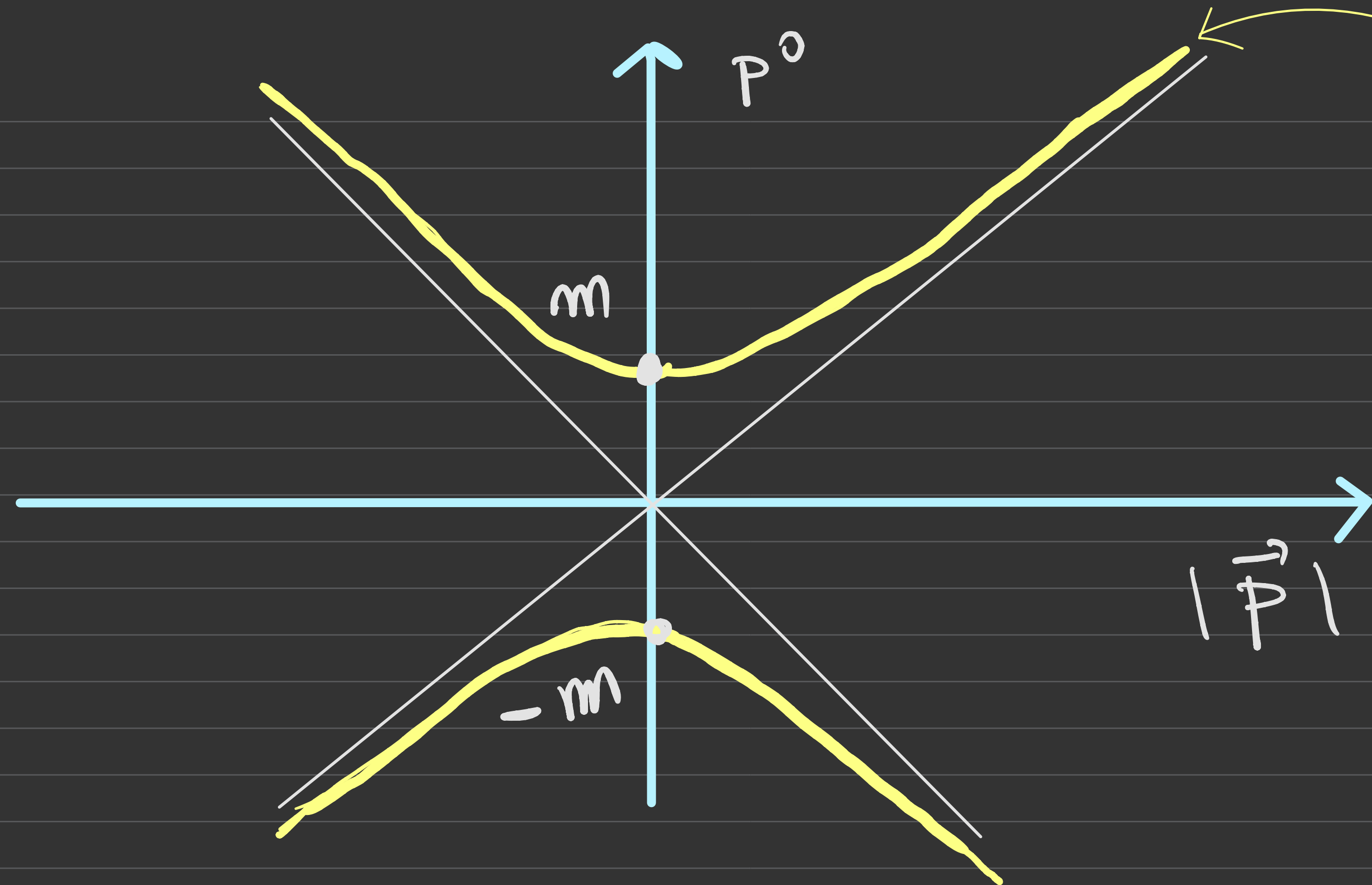
$$p^2 \equiv p^\mu p_\mu$$

- $$\hat{\psi}(\hat{p}) \neq 0 \implies \det(\not{p}-m) = (p^2-m^2)^2 = 0$$

$$\not{p} \equiv p_\mu \gamma^\mu \qquad \boxed{p^2 = m^2} \qquad \not{p}\not{p} = p^2$$

- $$(\not{p}+m)(\not{p}-m)\hat{\psi} = (\not{p}^2 + m\not{p} - \not{p}m - m^2)\hat{\psi} = (p^2-m^2)\hat{\psi} = 0$$

$$\implies \hat{\psi}(p) \propto \delta(p^2-m^2)$$



$$\boxed{p^2 = m^2 \quad p^0 = \omega_{\vec{p}} = \sqrt{m^2 + \vec{p}^{\,2}} > 0}$$

$$p^2 = m^2 \Rightarrow \quad p_0^2 - \vec{p}^{\,2} = m^2$$

$$p_0^2 = m^2 + \vec{p}^{\,2}$$

$$\boxed{p^2 = m^2 \quad p^0 = -\omega_{\vec{p}} = -\sqrt{m^2 + \vec{p}^{\,2}} < 0}$$

$$\hat{\psi}(p) = 2\pi\, \delta(p^2 - m^2) \Big(\theta(p^0)\, u(\vec{p}) + \theta(-p^0)\, v(-\vec{p}) \Big)$$

$$\Rightarrow \psi(x) = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\, \delta(p^2 - m^2)\, e^{-ipx} \Big(\underbrace{\theta(p^0)}_{p^0 = \omega_p}\, u(\vec{p}) + \underbrace{\theta(-p^0)}_{p^0 = -\omega_p}\, v(-\vec{p}) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \Big[e^{-i(\underbrace{\omega_p t - \vec{p}\cdot\vec{x}}_{\equiv p_\mu x^\mu})} u(\vec{p}) + e^{i(\omega_p t - \vec{p}\cdot\vec{x})} v(+\vec{p}) \Big]$$

$p^\mu = (\omega_p, \vec{p})$

$p_\mu = (\omega_p, -\vec{p})$

$$= \int d\mathcal{L}_I \Big[e^{-ipx} u(\vec{p}) + e^{ipx} v(\vec{p}) \Big]$$

- Goal: find most general $u(p)$, $v(p)$

$\equiv$ find a complete basis $u_i(p)$, $v_i(p)$ of solutions

$$u(p) = \sum_i a_i u_i(p)$$

$$v(p) = \sum_i b_i v_i(p)$$

- Positive energy solutions : $u(p)$

$$(\not{p}-m)\,u(p)=0 \qquad p^\mu=(\omega_p\,,\vec{p}\,)$$

- "rest frame": $p=(m,\vec{0}\,)\equiv\bar{p} \Rightarrow \not{p}\equiv p^\mu\gamma_\mu\equiv m\,\gamma_0=m\gamma^0$

$$(m\gamma^0-m)\,u(\bar{p})=m\begin{pmatrix}-\mathbb{1} & \mathbb{1}\\ \mathbb{1} & -\mathbb{1}\end{pmatrix}u(\bar{p})=0$$

$$u=\begin{pmatrix}u_L\\ u_R\end{pmatrix} \longrightarrow \begin{matrix}-u_L+u_R=0\\ u_L-u_R=0\end{matrix}$$

$$\Longrightarrow \quad u_L(\bar{p})=u_R(\bar{p})\equiv\sqrt{m}\,\xi \qquad \xi = \text{2D spinor}$$

- $u(\bar{p}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \Longrightarrow$ space of solutions $\equiv$ 2 component spinor

- rotations : $\xi \longrightarrow e^{-i\theta\cdot\sigma/2}\,\xi$ like spin $1/2$ wave function

- $\xi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv a_1 \xi_1 + a_2 \xi_2$

- $u(\bar{p}) = a_1 \sqrt{m} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} + a_2 \sqrt{m} \begin{pmatrix} \xi_2 \\ \xi_2 \end{pmatrix} \equiv a_1 u_1 + a_2 u_2$

$= a_1 \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_2 \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

- Solution for arbitrary $p^\mu = (\omega_p, \vec{p})$

- <u>idea</u>: same as case of $\bar{p}^\mu = (m, 0)$, but viewed in a moving reference frame

- $p^\mu = \Lambda_p{}^\mu{}_\nu \, \bar{p}^\nu$
$$\begin{cases} \Lambda_p = e^{i\vec{\eta}_p \cdot \vec{K}} \equiv \Lambda(\eta) \\ \vec{\eta}_p = \dfrac{\vec{p}}{|p|} \tanh^{-1}\left(\dfrac{|p|}{\omega_p}\right) \end{cases}$$

- <u>claim</u>: $u(p) \equiv \Lambda_D(\vec{\eta}_p)\, u(\bar{p})$ satisfies $(\not{p} - m)\, u(p) = 0$

$$\Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

• Notice $\Lambda_D \cancel{K} \Lambda_D^{-1} = (\Lambda^{\mu}{}_{\nu} K^{\nu})\gamma_\mu \equiv \cancel{(\Lambda K)}$

$K^\mu \gamma_\mu$

θ, η

$$(\bar{\cancel{p}} - m)\, u(\bar{p}) = 0 \implies$$

$$0 = \Lambda_D (\bar{\cancel{p}} - m)\, u(\bar{p}) = \Lambda_D (\bar{\cancel{p}} - m) \Lambda_D^{-1} \Lambda_D\, u(\bar{p})$$

$$= \left[(\Lambda \bar{p})^\mu \gamma_\mu - m\right] u(p)$$

$$= (\cancel{p} - m)\, u(p)$$

Concretely one finds (Exercise 1)

$$u(\bar{p}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \implies u(p) = \Lambda_D \, u(\bar{p}) = \begin{pmatrix} \sqrt{p^\mu \sigma_\mu} \; \xi \\ \sqrt{p^\mu \bar{\sigma}_\mu} \; \xi \end{pmatrix}$$

- $$\sqrt{p^\mu \sigma_\mu} \equiv \sqrt{p \cdot \sigma} = \frac{1}{\sqrt{2}} \left(\sqrt{\omega_p + m} - \frac{\vec{p} \cdot \vec{\sigma}}{\sqrt{\omega_p + m}} \right)$$

- $$\sqrt{p^\mu \bar{\sigma}_\mu} \equiv \sqrt{p \cdot \bar{\sigma}} = \frac{1}{\sqrt{2}} \left(\sqrt{\omega_p + m} + \frac{\vec{p} \cdot \vec{\sigma}}{\sqrt{\omega_p + m}} \right)$$

$$\left(\sqrt{p\cdot\sigma}\right)^2 = \left[\frac{1}{\sqrt{2}}\left(\sqrt{\omega_p+m} - \frac{\vec{p}\cdot\vec{\sigma}}{\sqrt{\omega_p+m}}\right)\right]^2$$

$$= \frac{1}{2}\left(\omega_p+m + \frac{(\vec{p}\cdot\vec{\sigma})^2}{\omega_p+m} - 2\,\vec{p}\cdot\vec{\sigma}\right)$$

$$(\vec{p}\cdot\vec{\sigma})^2 = |\vec{p}|^2$$

$$= \frac{1}{2}\left(\omega_p+m + \frac{|\vec{p}|^2}{\omega_p+m} - 2\vec{p}\cdot\vec{\sigma}\right)$$

$$\sigma^i\sigma^j+\sigma^j\sigma^i = 2\delta^{ij}$$

$$= \frac{1}{2}\,\frac{(\omega_p+m)^2+\vec{p}^{\,2}}{\omega_p+m} - \vec{p}\cdot\vec{\sigma}$$

$$= \frac{1}{2}\,\frac{\omega_p^2 + 2\omega_p m + m^2 + \vec{p}^{\,2}}{\omega_p + m} - \vec{p}\cdot\vec{\sigma} \qquad \omega_p^2 = m^2 + \vec{p}^{\,2}$$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}\omega_p^2 + \cancel{2}\omega_p m}{\omega_p + m} - \vec{p}\cdot\vec{\sigma}$$

$$= \frac{\omega_p(\omega_p + m)}{\omega_p + m} - \vec{p}\cdot\vec{\sigma}$$

$$= \omega_p - \vec{p}\cdot\vec{\sigma} = p^\mu \sigma_\mu$$

$$\sqrt{p\cdot\sigma}\,\sqrt{p\cdot\bar{\sigma}} = \sqrt{p\cdot\bar{\sigma}}\,\sqrt{p\cdot\sigma} = \sqrt{p^2} = m$$

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix}\begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\,\xi \\ \sqrt{p^\mu\bar{\sigma}_\mu}\,\xi \end{pmatrix} = \left(-m\sqrt{p\sigma} + p\cdot\sigma\sqrt{p\cdot\bar{\sigma}}\right)\xi$$

Chirality & Helicity

- **Chirality**

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\left\{ \begin{array}{l} \psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ \gamma_5 \psi_D = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix} \end{array} \right.$$

$$P_L = \left(\frac{1-\gamma_5}{2} \right) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad P_R = \left(\frac{\mathbb{1}+\gamma_5}{2} \right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \sim \left(\tfrac{1}{2}, 0\right) \qquad P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \sim \left(0, \tfrac{1}{2}\right)$$

- solutions of Dirac eq. are <u>not</u> eigenstates of γ_5

  "<u>not</u> chiral"

- $m \to 0$, ψ_L and ψ_R decoupled $\Rightarrow$ expect chiral solutions

- intuitively: $m \to 0 \sim \omega_p \to \infty$

Helicity: $\hat{h} \equiv \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|} = \frac{\vec{p} \cdot \vec{\sigma}}{2|\vec{p}|}$ $\qquad \hat{h}^2 = \frac{\mathbb{1}}{4}$

check as exercise

$m \to 0$: $\qquad u(p) = \sqrt{\frac{\omega_p}{2}} \begin{pmatrix} (1-2\hat{h})\xi \\ (1+2\hat{h})\xi \end{pmatrix}$

$$\begin{pmatrix} \hat{h} & 0 \\ 0 & \hat{h} \end{pmatrix} u(p) = \sqrt{\frac{\omega_p}{2}} \begin{pmatrix} (\hat{h}-\frac{1}{2})\xi \\ (\hat{h}+\frac{1}{2})\xi \end{pmatrix} = \begin{pmatrix} -\frac{\mathbb{1}}{2} & 0 \\ 0 & \frac{\mathbb{1}}{2} \end{pmatrix} u(p)$$

if $p = (0, 0, |p|)$

$\hat{h} = \frac{\sigma_3}{2}$ $\qquad \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & +\mathbb{1} \end{pmatrix}$$

$\bullet \; \xi_+ , \xi_- \; \Big| \; \hat{h}\,\xi_\pm = \pm \frac{1}{2} \xi_\pm \implies$

$$u_- = \sqrt{\frac{\omega_p}{2}} \begin{pmatrix} \xi_- \\ 0 \end{pmatrix} \quad \text{Left}$$

$$u_+ = \sqrt{\frac{\omega_p}{2}} \begin{pmatrix} 0 \\ \xi_+ \end{pmatrix} \quad \text{Right}$$

- expectedly, chiral solutions exist

$m = 0$

- chirality eigenstate $\Longleftrightarrow$ helicity eigenstate

$u_- \longrightarrow$ left spinor with $\hat{h} = -\frac{1}{2}$

$u_+ \longrightarrow$ right spinor with $\hat{h} = +\frac{1}{2}$

Normalization

$$u_r(\vec{p}) = \begin{pmatrix} \sqrt{p\sigma}\, \xi_r \\ \sqrt{p\bar{\sigma}}\, \xi_r \end{pmatrix} \qquad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$p^\mu = (\omega_p, \vec{p})$$

- $$\bar{u}_r u_s = u_r^+ \gamma_0 u_s = \left(\xi_r^+ \sqrt{p\sigma},\ \xi_r^+ \sqrt{p\bar{\sigma}}\right) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\sigma}\, \xi_s \\ \sqrt{p\bar{\sigma}}\, \xi_s \end{pmatrix}$$

$$= \xi_r^+ \left(\sqrt{p\sigma}\sqrt{p\bar{\sigma}} + \sqrt{p\bar{\sigma}}\sqrt{p\sigma} \right) \xi_s = 2m\, \xi_r^+ \xi_s = 2m\, \delta_{rs}$$

$$= \sqrt{(p\sigma)(p\bar{\sigma})} = \sqrt{p^2} = m$$

- $\bar{u}_s \gamma^\mu u_r = 2p^\mu \delta_{rs}$

<u>Exercise 3</u>

$$\psi(x) = \int d\Omega_I \left[e^{-ipx} u(p) + e^{ipx} v(p) \right]$$

$$\sum_{r=1}^{2} a_r u_r(\vec{p})$$

$$\downarrow$$

$$u_r(\vec{p}) = \begin{pmatrix} \sqrt{p\sigma}\, \xi_r \\ \sqrt{p\bar{\sigma}}_r \end{pmatrix}$$

$$(\not{p} + m) v(\vec{p}) = 0$$

$$v_r(\vec{p}) = \begin{pmatrix} \sqrt{p\sigma}\ \xi_r \\ -\sqrt{p\bar{\sigma}}\ \xi_r \end{pmatrix}$$

<u>Exercise 3</u> :

- $\bar{v}_r(\vec{p})\, v_s(\vec{p}) = -2m\,\delta_{rs}$
- $\bar{v}_r(\vec{p})\,\gamma^\mu v_s(\vec{p}) = 2p^\mu\,\delta_{rs}$
- $\bar{u}_r(\vec{p})\, v_s(\vec{p}) = \bar{v}_r(\vec{p})\, u_s(\vec{p}) = 0$
- $u_r^\dagger(\vec{p})\, v_s(-\vec{p}) = v_r^\dagger(-\vec{p})\, u_s(\vec{p}) = 0$

$\triangle$ Completeness relations

$\sum_i |i\rangle\langle i| = \mathbb{1}$

$$\sum_{r=1}^{2} \xi_r \xi_r^{+} = \mathbb{1}_{2\times 2}$$

$$u_r(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_r \\ \sqrt{p\bar{\sigma}}\,\xi_r \end{pmatrix}$$

$$v_r(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_r \\ -\sqrt{p\bar{\sigma}}\,\xi_r \end{pmatrix}$$

$$\sum_s u_s \bar{u}_s = \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} = \not{p} + m$$

$$\sum_s v_s \bar{v}_s = \begin{pmatrix} -m & p\sigma \\ p\bar{\sigma} & -m \end{pmatrix} = \not{p} - m$$